- → Need identify characteristic $\begin{cases} \text{collective} \\ \text{collisional} \end{cases}$ scales

→ plasma is continuum ⇒ characterize by collective modes (can calculate response).

II) Plasma/Fluid Collective Modes, Response

why $\epsilon = 0$

a.) Cold Plasma ($T, P \to 0$)

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{v}) = 0 \quad \to \text{continuity}$$

$$nm \frac{d\underline{v}}{dt} = n\left(q\underline{E} + \frac{\underline{v}}{c} \times \underline{B}\right) \quad \to \text{momentum balance}$$

+ Maxwell Equations

For electromagnetic/electrostatic wave:

$$n = n_0 + \tilde{n}$$
$$\underline{v} = \cancel{\underline{v}_0} + \tilde{\underline{v}}$$
$$\underline{E} = \cancel{\underline{E}_0} + \tilde{\underline{E}}$$
$$\underline{B} = \cancel{\underline{B}_0} + \tilde{\underline{B}}$$

$\Rightarrow$ 1 species - ions stationary

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \nabla \cdot \tilde{\underline{V}}$$

$$\frac{\partial \tilde{\underline{V}}}{\partial t} = \frac{q}{m} \tilde{\underline{E}} + \cancel{\frac{q\, \tilde{\underline{V}} \times \tilde{\underline{B}}}{mc}}$$

$$\nabla \cdot \tilde{\underline{E}} = 4\pi q \tilde{n} \qquad \nabla \cdot \tilde{\underline{B}} = 0$$

$$\nabla \times \tilde{\underline{B}} = \frac{4\pi}{c} \tilde{\underline{J}} + \frac{1}{c} \frac{\partial \tilde{\underline{E}}}{\partial t} \qquad \nabla \times \tilde{\underline{E}} = -\frac{1}{c} \frac{\partial \tilde{\underline{B}}}{\partial t}$$

$$\tilde{\underline{J}} = n_0 q \tilde{\underline{V}}$$

$\Rightarrow$ Fourier Transforming $\qquad \underset{etc}{\underline{E}} = \sum_{\underline{k},\omega} \underline{E}_{\underline{k},\omega} e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

$$\underline{k} \times \tilde{\underline{B}}_{\underline{k},\omega} = -\frac{4\pi n_0}{c} q i \tilde{\underline{V}}_{\underline{k},\omega} - \frac{\omega}{c} \tilde{\underline{E}}_{\underline{k},\omega}$$

$$\frac{\underline{k}}{\omega} \times (\underline{k} \times \underline{E}_{\underline{k},\omega}) = \frac{4\pi n_0 q^2}{m_e \omega} \underline{E}_{\underline{k},\omega} - \frac{\omega}{c} \underline{E}_{\underline{k},\omega}$$

$\Rightarrow$

$$\underline{k}(\underline{k} \cdot \underline{E}_{\underline{k},\omega}) - k^2 \underline{E}_{\underline{k},\omega} = \underbrace{\frac{4\pi n_0 q^2}{c^2 m_e}}_{\omega_p^2/c^2} \underline{E}_{\underline{k},\omega} - \frac{\omega^2}{c^2} \underline{E}_{\underline{k},\omega}$$

→ EM waves $(\underline{k}\cdot\underline{E}_{k\omega}=0)$

$$k^2\underline{E}_{k\omega}-\underline{k}(\underline{k}\cdot\underline{E}_{k\omega}) = \frac{\omega^2}{c^2}\underline{E}_{k,\omega}-\frac{\omega_p^2}{c^2}\underline{E}_{k\omega}$$

$\omega_p^2 = 4\pi n_0 q^2/m$ → plasma frequency
→ characteristic frequency for (non-neutralized) plasma oscillations
→ (ions stationary ⇒ $\omega \gg \omega_{pi}$ $(\sim 1/M_i)$)

$$k^2 = \frac{\omega^2}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)$$

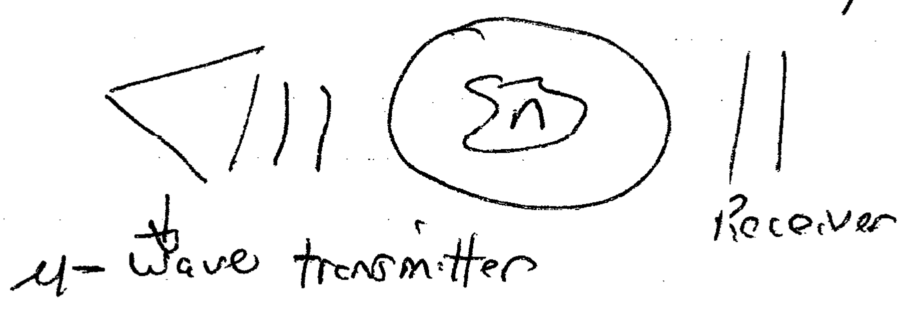
$$\omega^2 = \omega_p^2 + c^2k^2$$

Dispersion Relation for EM Waves in Unmagnetized plasma

→ $\epsilon(\omega) = 1-\omega_p^2/\omega^2$ — cold plasma dielectric (dispersive)

→ $\omega<\omega_p \Rightarrow k^2<0$ — ω_p is cut-off frequency

→ can diagnose density



→ Electrostatic Waves / oscillations (Langmuir Osc.)

$\underline{k} \cdot \underline{E} = k\underline{E}$ ↔ alternatively obtain via $\left\{\begin{array}{l}\text{Fluid eqns} \\ + \\ \text{Gauss Law}\end{array}\right.$

$$\Rightarrow \quad 0 = \left[(\omega^2 - \omega_p^2)/c^2\right] \underline{E}_{k,0}$$

$$\omega^2 = \omega_p^2$$

- ions stationary ↔ $\omega^2 \gg \omega_{pi}^2 \sim 1/M_i$
- non-propagating oscillation $\omega^2 = \omega_p^2$

b.) Warm Plasma Waves (Electrostatic) (Langmuir Waves)

Now, introduce pressure

$$m \frac{\partial \tilde{\underline{V}}}{\partial t} = q\,\tilde{\underline{E}} - \frac{\underline{\nabla}\,\tilde{p}}{n_0}$$

$$\frac{\partial \tilde{n}}{\partial t} = -n_0\, \underline{\nabla} \cdot \underline{V}$$

$$\underline{\nabla} \cdot \tilde{\underline{E}} = 4\pi q\, \tilde{n}$$

$\left\{\begin{array}{l} p = p_0 (n/n_0)^\gamma \\ \quad - \text{adiabatic} \\ p = \tilde{n}\, T \\ \quad - \text{isothermal} \end{array}\right.$

↓

determine eqn. state from kin. Th.

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = -n_0 \left(\frac{q}{m} \nabla \cdot \underline{E} - \frac{\nabla^2 \tilde{p}}{n_0 m} \right)$$

$$= -\omega_p^2 \tilde{n} + \frac{T}{m} \nabla^2 \tilde{n}$$

$$\Rightarrow \frac{\partial^2 \tilde{n}}{\partial t^2} = \underbrace{-\omega_p^2 \tilde{n}}_{\text{plasma oscillation}} + \underbrace{\frac{T}{m} \nabla^2 \tilde{n}}_{\text{streaming induced by } \nabla p \text{ (a la' acoustics)}} \qquad \frac{T}{m} \equiv V_{The}^2$$

$$\omega^2 = \omega_p^2 + k^2 V_{Th}^2$$

$$\rightarrow \omega^2 = \omega_p^2 (1 + k^2 \lambda_D^2)$$

$$\lambda_D^2 \equiv V_{Th}^2 / \omega_p^2 \quad \rightarrow \text{Debye Length}$$

i.e.

$$\nabla^2 \tilde{n} - \frac{1}{\lambda_D^2} \tilde{n} = \frac{1}{V_T^2} \frac{\partial^2 \tilde{n}}{\partial t^2} \Rightarrow \omega \to 0 \text{ recovers screened Gauss' Law}$$

Recall Debye Length !

$$\nabla^2 \phi = 4\pi \rho_{ind} + 4\pi q\, \delta(\underline{x} - \underline{x}_0)$$

screening response ... $\rightarrow$ test charge in plasma

$$f = n_0 \exp\left[-\frac{mv^2}{T} \pm \frac{q\phi}{T}\right]$$

$$\rho_{ind} = n_0 q \exp[q\phi/T_e] - n_0 q \exp[-q\phi/T_i]$$

$$\cong \frac{\omega_{pe}^2}{4\pi V_{Te}^2}\phi + \frac{\omega_{pi}^2}{4\pi V_{Ti}^2}\phi$$

$$\Rightarrow \nabla^2\phi - \left(\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right)\phi = 4\pi q\,\delta(\underline{x} - \underline{x}_0)$$

Then $\omega \ll \omega_p$ ⇒ plasma response is streaming to screen test charge

⇒ hence appearance Debye length

$\omega \gg \omega_p$ ⇒ warm plasma oscillation (too fast to screen)

Note: cold plasma $(T=0)$ ⇒ no energy to move to screen charge

⇒ Warm Plasma Wave combines $\left\{\begin{array}{l}\text{plasma oscillation} \\ \oplus \text{ acoustic wave}\end{array}\right.$

i.e. carries wave momentum.

·) Ion Acoustic Wave

– so far, 'single species' dynamics

– consider now, ion acoustic wave, with
$V_{Ti} < \frac{\omega}{k} < V_{Te}$

Recall, for warm electrons:

$$\frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\cancel{\tilde{n}} \tilde{V}) = -n_0 \nabla \cdot \underline{\tilde{V}}$$

$$m_e \frac{\partial \tilde{V}}{\partial t} = -|e| \tilde{E} - T_e \frac{\nabla \tilde{n}}{n_0} \qquad (\tilde{p} :: \tilde{n}\, T_0)$$

$$= +|e| \nabla \tilde{\phi} - T_e \frac{\nabla \tilde{n}}{n_0}$$

$\Rightarrow$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = -n_0 \left(\frac{|e|}{m_e} \nabla^2 \tilde{\phi} - \frac{V_{Te}^2}{n_0} \nabla^2 \tilde{n} \right)$$

$$\underbrace{\frac{\partial^2 \tilde{n}}{\partial t^2}}_{O(\omega^2)} - \underbrace{V_{Te}^2 \nabla^2 \tilde{n}}_{\substack{\text{electron compression} \\ O(k^2 V_{Te}^2)}} = -n_0 \frac{|e|}{m_e} \nabla^2 \tilde{\phi}$$

$\therefore$ for $k^2 V_{Te}^2 \gg \omega^2$

$$\frac{\tilde{n}}{n_0} = \frac{|e|\tilde{\phi}}{T_e}$$

Note:

→ equivalent to limit where electron inertia negligible

i.e. $m_e \to 0$ $\quad (V_{Te}^2 \to \infty) \Rightarrow \dfrac{\tilde{n}}{n} = \dfrac{|e|\tilde{\phi}}{T_e}$

→ could, in limit $k^2 V_{Te}^2 \gg \omega^2$, obtain from Boltzmann response

i.e. $\quad E \to E - |e|\tilde{\phi} \quad \Rightarrow \quad f_e = c\exp\left[-\dfrac{\left(\frac{mv^2}{2} - |e|\tilde{\phi}\right)}{T}\right]$

$$\cong \left(1 + \frac{|e|\tilde{\phi}}{T}\right) f_{0M}$$

For ions (cold)

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \, \underline{\nabla} \cdot \underline{\tilde{v}}$$

$$\frac{\partial \tilde{v}_i}{\partial t} = +\frac{|e|}{m_i} \tilde{E}$$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = +n_0 \frac{|e|}{m_i} \nabla^2 \tilde{\phi}$$

$$\frac{\hat{n}_{i\,k,\omega}}{n_0} = + \frac{|e|}{m_i} \frac{k^2}{\omega^2} \tilde{\phi}_{k,\omega}$$

$$\nabla^2 \phi = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$\cdot k^2 \hat{\phi}_{k,\omega} = +4\pi n_0 |e| \left(\frac{|e|}{m_i} \frac{k^2}{\omega^2} \hat{\phi}_{k,\omega} - \frac{|e| \hat{\phi}_{k,\omega}}{T_e} \right)$$

$$k^2 = \frac{\omega_{pi}^2 k^2}{\omega^2} - \underbrace{\frac{\omega_{pe}^2}{V_{Te}^2}}_{\to \lambda_{De}}$$

$$\Rightarrow \left(k^2 + 1/\lambda_{De}^2 \right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\therefore \boxed{\begin{aligned} \omega^2 &= k^2 c_s^2 \Big/ \left(1 + k^2 \lambda_{De}^2\right) \\ c_s^2 &= T_e / m_i \end{aligned}}$$

Note:

→ Compare hydrodynamic acoustic wave:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \nabla \cdot \tilde{V} \quad ; \quad \frac{\partial \tilde{V}}{\partial t} = -\frac{\nabla \tilde{p}}{\rho_0}$$

$$\tilde{p} = c_s^2 \tilde{\rho}$$

	Hydro	Ion-Acoustic
"Springiness"	Gas Pressure	T_e
Inertia	Gas Density/inertia	m_i

i.e. ion-acoustic wave is two component, hybrid oscillation

$$\rightarrow \left(k^2 + 1/\lambda_{De}^2\right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\left(1 + 1/k^2\lambda_{De}^2\right) = \frac{\omega_{pi}^2}{\omega^2}$$

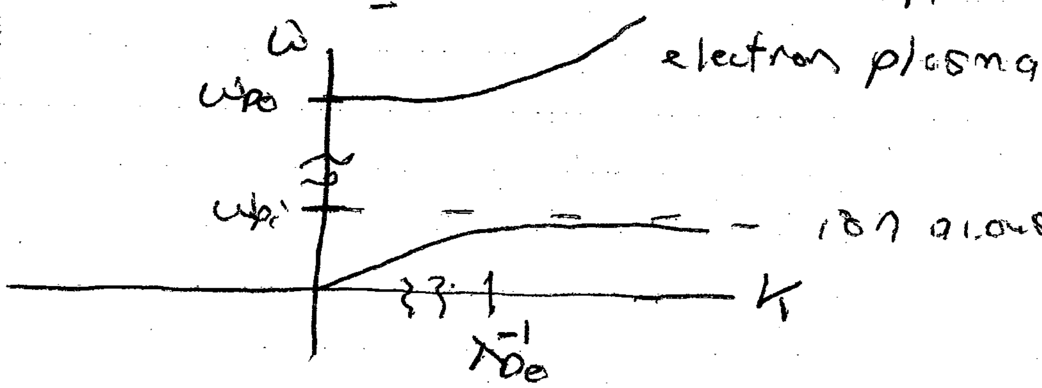
Debye shielding (T_e)

$\rightarrow$ ion plasma oscillation (m_i)

Ion-acoustic wave as Debye-shielded ion plasma oscillation

Note : $k^2 \lambda_{De}^2 \gtrsim 1 \Rightarrow \omega^2 \to \omega_{pi}^2$

$\Rightarrow$



Basic modes (electrostatic) of un-magnetized plasma.

Basic Scales : $\left\{ \begin{array}{l} \omega_{pe},\ \omega_{pi} \\ \lambda_{De} \\ V_{Te},\ C_s \end{array} \right.$

c.) Nonlinear Fluid Plasma Waves

$\Rightarrow$ Langmuir, Ion Acoustic Waves $\Rightarrow$ 1D
Compressional Waves

$\Rightarrow$ 1D Compressional Wave (Linear)

$\downarrow$ (steepening – finite amplitude)

Shock

collisional (standard) (Burgers) $\leftarrow$ Shock $\rightarrow$ collisionless (KdV)

→ Phase space flow incompressible (Liouville Thm.)

→ Derive Vlasov Eqn. from:

- Liouville Eqn.
- $N = \sum_i \delta(\underline{x}-\underline{x}_i)\,\delta(\underline{v}-\underline{v}_i)$ "peas" } Klimontovich Eqn.
- hierarchy, with $f(\underline{X}_1, \underline{X}_2, t) =$ "crushed pea soup" ← $f(\underline{X}_1, t)\,f(\underline{X}_2, t) + g(\underline{X}_1, \underline{X}_2, t)$

  and $1/n x_0^3 \ll 1 \Rightarrow g \ll f^2$ etc.

(Return in Fluctuations Discussion).

IV.) Collective Response in Collisionless Plasma

→ Waves in Vlasov Plasma (1D)

- $\omega, kV \gg \nu \Rightarrow$

  $f = \langle f \rangle + \tilde{f}$

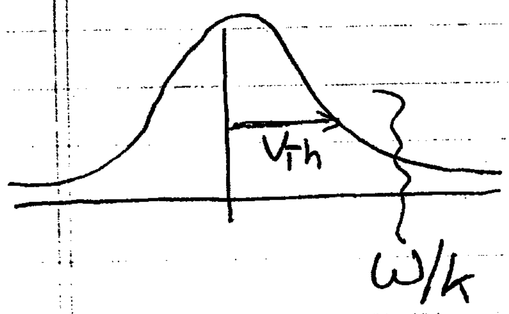
  $\langle f \rangle = (1/\sqrt{2\pi}\,V_{Th})\exp(-V^2/2V_{Th}^2)$ (Maxwellian)

i.e. $\langle f \rangle$ established on long-time scale

– Seek contact with Langmuir Wave (ions stationary)

$\Rightarrow \omega > k v_{Th}$

(Heuristic)



Then, linearize:

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\nabla^2 \tilde{\phi} = -4\pi n_0 q \int \tilde{f}\, dv$$

$$f = \sum_{k,\omega} f_{k,\omega}\, e^{i(kx - \omega t)}$$

$$\Rightarrow \quad -i(\omega - kv)\, \tilde{f}_{k,\omega} = \frac{q}{m}\, ik \tilde{\phi}_{k,\omega}\, \partial \langle f \rangle / \partial v$$

$$+ k^2 \tilde{\phi}_{k,\omega} = 4\pi n_0 q \int \tilde{f}_{k,\omega}\, dv$$

$$\tilde{f}_{k,\omega} = -k \frac{q}{m} \frac{\tilde{\phi}_{k,\omega}\, \partial \langle f \rangle / \partial v}{(\omega - kv)}$$

so $$k^2 \tilde{\phi}_{k,\omega} = -\omega_p^2 k \int dv\, \frac{\partial \langle f \rangle / \partial v}{(\omega - kv)}\, \tilde{\phi}_{k,\omega}$$

Thus,
$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial\langle f\rangle/\partial v}{(\omega - kv)}$$

– dielectric function for Vlasov Plasma

$\boxed{\text{? How Handle Pole at } \omega = kv \text{ ?}}$

Concepts
– wave-particle resonance
– collisionless damping

– Recall V.E. derived in limit $\gamma \to 0$

$$1/\omega - kv = \lim_{\epsilon \to 0} 1/\omega - kv + i\epsilon$$

– Alternatively, causality requires: $\tilde{\phi} \underset{t \to -\infty}{\to} 0$

$$\phi \sim e^{-i\omega t} \Rightarrow \phi \sim e^{-i(\omega + i\epsilon)t}$$

(i.e. formally IVP)

$\therefore$

$$1/\omega - kv = \lim_{\epsilon \to 0} 1/\omega - kv + i\epsilon$$

$$= \frac{P}{\omega - kv} - i\pi\delta(\omega - kv)$$

(Plemelj Formulae)

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$$

$$= 1 + \frac{\omega_p^2}{k}\int dv \, \frac{P}{\omega - kv}\frac{\partial\langle f\rangle}{\partial v}$$

$$-i\pi \frac{\omega_p^2}{k|k|}\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k} \rightarrow \text{physical content!?}$$

i.e.

$$\delta(\omega - kv) = \frac{1}{|k|}\delta(v - \omega/k)$$

Further: $\frac{\partial\langle f\rangle}{\partial v} = -\frac{v}{V_{Th}}\langle f\rangle$

$$kV_{Th} < \omega \Rightarrow \frac{P}{\omega - kv} = \frac{1}{\omega}\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \dots\right)$$

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{kV_{Th}^2}\int \frac{\langle f\rangle v}{\omega}\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \dots\right)$$

$$= 1 - \frac{\omega_p^2}{\omega^2} - 3\omega_p^2\frac{V_{Th}^2 k^2}{\omega^4}$$

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

so

$$\epsilon = \epsilon_R + i\,\epsilon_{IM}$$

$$\epsilon_R = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

$$\epsilon_{IM} = -\pi\frac{\omega_p^2}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}$$

$\rightarrow \epsilon_R = 0 \Rightarrow$ Collective Resonance / Wave

- as ϵ derived via $(kv/\omega) < 1$ expansion, need determine $\omega(k)$ iteratively

$$\epsilon_r = 0 = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

Lowest order: $\omega^{(0)} = \omega_p$

$$\Rightarrow \quad \epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega_p^2}\right)$$

$\therefore \quad \omega^2 = \omega_p^2(1 + 3k^2\lambda_D^2) \longrightarrow$ structure agrees with fluid mdl.

$\hookrightarrow$ contrast fluid

— Distribution function determines equation of state

i.e. # 3 $\leftrightarrow \int v^4 \langle f \rangle$

Contrast K + T : $p = p_0 (\rho/\rho_0)^\gamma$ $\gamma = 3$

$\gamma = 3 \leftrightarrow$ Maxwellian

— structure of dispersion relation identical to warm fluid model $\leftrightarrow k v_{Th} < \omega$.

$\rightarrow \epsilon_{IM}$.

$$\epsilon_{IM} = -\pi \frac{\omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\Big|_{\omega/k}$$

$$Q = \omega\, \epsilon_{IM} \left(|E|^2/8\pi\right)$$ $\rightarrow$ dissipated energy

$$\Rightarrow \quad Q = -\omega_k \frac{\pi \omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\Big|_{\omega_k/k} |E|^2/8\pi$$

Now,
$$\frac{\partial W_{\underline{k}}}{\partial t} + \nabla \cdot S_{\underline{k}} + Q_{\underline{k}} = 0$$

$$\Rightarrow \gamma_{\underline{k}} = -Q_{\underline{k}}/W_{\underline{k}} \qquad W_{\underline{k}} = \omega_{\underline{k}} \left.\frac{\partial \epsilon_r}{\partial \omega}\right|_{\omega_{\underline{k}}} \frac{|E|^2}{8\pi}$$

$$\therefore \quad \gamma_{\underline{k}} = \left(\frac{\pi \omega_p^2}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\frac{\omega_{\underline{k}}}{k}}\right) \Big/ \left(\left.\frac{\partial \epsilon_r}{\partial \omega}\right|_{\omega_{\underline{k}}}\right)$$

Alternatively:

$$\epsilon = \epsilon_R(\underline{k}, \omega) + i\,\epsilon_{IM}(\underline{k}, \omega)$$

$$\omega = \omega_{\underline{k}} + i\gamma_{\underline{k}} \qquad \gamma \ll \omega_{\underline{k}}$$

$$\epsilon = \epsilon_R(k, \omega_k + i\gamma_k) + i\,\epsilon_{IM}(k, \omega_{\underline{k}})$$
$$\cong \epsilon_R(k, \omega_{\underline{k}}) + i\gamma_{\underline{k}} \left.\frac{\partial \epsilon_R}{\partial \omega}\right|_{\omega_{\underline{k}}} + i\,\epsilon_{IM}(k, \omega_{\underline{k}})$$

$$\gamma_{\underline{k}} = -\epsilon_{IM}(k, \omega_{\underline{k}}) \Big/ (\partial \epsilon_R/\partial \omega)\Big|_{\omega_{\underline{k}}}$$

agrees above.

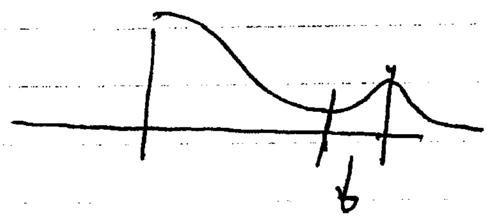
Thus → $\partial\langle f\rangle/\partial v\big|_{\omega_k/k} < 0$

$\Rightarrow$ damping (Landau damping)

→ $\partial\langle f\rangle/\partial v\big|_{\omega/k} > 0$

$\Rightarrow$ growth

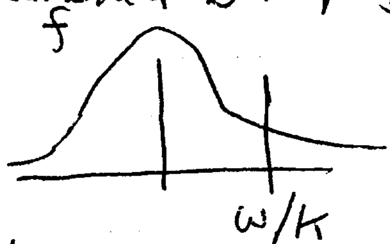
i.e. 'Bump on Tail'



$\omega/k \sim v$ grows as $\partial\langle f\rangle/\partial v > 0$

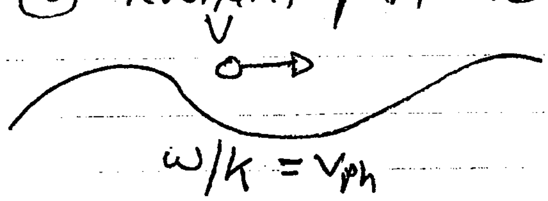
→ Physics of Landau Damping

Consider



→ Landau damping occurs due wave particle resonance $\omega/k \sim v$

→ intuitively, consider wave interaction with ~ resonant particle



Resonant particle 'sees' ~ DC field

$$\frac{dv}{dt} = \frac{q}{m} E \cos(kx-\omega t)$$

$$= \frac{q}{m} E \cos\left(k\left(x - v_{ph}t\right)\right)$$

if boost to frame at particle velocity V

$x' = x - Vt$
$v' = v - V$
$a' = a$

$\Rightarrow$

$$\frac{dv}{dt} = \frac{q}{m} E \cos\left(k\left(x + (V - v_{ph})t\right)\right)$$

$\therefore$

- secular (in time) interaction at $V \sim V_{ph}$ resonance
- $v \lesssim \omega/k \Rightarrow$ wave accelerates particles, loses energy

  $v \gtrsim \omega/k \Rightarrow$ wave decelerates particles, gains energy

Q = # accelerated − # decelerated

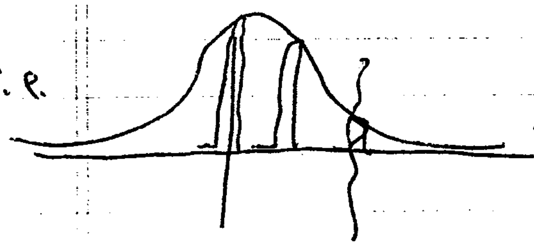
$\sim \left.(\partial f/\partial v)\right|_{\omega/k}$

▷ Quantitatively :

– as $Q = \langle \underline{E}^* \cdot \underline{J} \rangle$

seek $\bar{q} = \langle q v E \rangle$ ⇒ time averaged work on resonant 'beam'

i.e.



⇒ plasma distribution as superposition of beams

then $Q = \int dv\, \bar{q}$

– $v = v_0 + \delta v$ ↦ perturbations induced by wave

$x = x_0 + \delta x$

so
$$\frac{d\delta v}{dt} = \frac{q}{m} E \Big|_{x_0, v_0}$$

$$\frac{d\delta x}{dt} = \delta v$$

$\bar{q} = q \langle v E \rangle$

$v = v_0 + \delta v$

$E = E(t, x = x_0 + \delta x)$

$\cong E(t, x_0) + \delta x\, \partial E / \partial x \big|_{x_0}$

$$\bar{q} = q\left\langle \left(\underset{\text{DC}}{V_0} + \underset{\text{osc}}{\delta V}\right)\left(\underset{\text{osc}}{E(t,x_0)} + \underset{\text{both osc.}}{\delta x\, \partial E/\partial x\big|_{x_0 t}}\right)\right\rangle$$

$$\bar{q} = q V_0 \left\langle \delta x\, \partial E/\partial x\big|_{x_0 t} \right\rangle + q\left\langle \delta V\, E(t, x_0)\right\rangle$$

Now, $\quad \dfrac{d\delta V}{dt} = \dfrac{q}{m} E(t, x_0) \qquad\qquad x_0 = x_0' + V_0 t$

$$= \frac{q}{m} E_0\, e^{ikx_0'}\, e^{ik(V_0 - \omega/k)t}\, e^{\delta t}$$

$x_0' = 0$ (convenience)

$\omega/k = v_{ph}$

$\delta > 0 \Rightarrow \delta V \to 0$ as $t \to -\infty$

$\therefore$

$$\frac{d\delta V}{dt} = \frac{q}{m} E_0 \exp\left(ik\left(V_0 - \omega/k - i\delta\right)t\right)$$

$$\delta V = \frac{q}{m} \left.\frac{E_0\, e^{ik(V_0 - \omega/k - i\delta)t}}{i\left(k(V_0 - v_{ph}) - i\delta\right)}\right|_{-\infty}^{t}$$

$$\Rightarrow \quad \delta V = \frac{q}{m} E(t, x_0) \Big/ \left(ik(V_0 - v_{ph}) + \delta\right)$$

$$\delta x = \frac{q}{m} E(t\, x_0) \Big/ \left(ik(V_0 - v_{ph}) + \delta\right)^2$$

Thus

$$\bar{q} = q V_0 \left\langle \delta x \frac{\partial E}{\partial x} \right\rangle + q \left\langle \delta V \, E \right\rangle$$

$$= q V_0 \left\langle -ik\, E^*(t, x_0) \, \frac{q}{m} \frac{E(t, x_0)}{(ik(V_0 - V_p) + \delta)^2} \right\rangle$$

$$+ q \left\langle E^*(t, x_0) \, \frac{q}{m} \frac{E(t, x_0)}{(ik(V_0 - V_p) + \delta)} \right\rangle$$

note! $E^* E$ gives DC beat

$\Rightarrow$

$$\bar{q} = \frac{d}{dV_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{V_0}{ik(V_0 - V_p) + \delta} \right\}$$

$$= \frac{d}{dV_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{-i V_0}{k(V_0 - V_p) - i\delta} \right\}$$

note! '2' from $\cos^2$

real part $\Rightarrow$

$$\bar{q} = \frac{d}{dV_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{V_0 \pi}{|k|} \delta(V_0 - V_p) \right\}$$

$$Q = n\int dv_0\, \bar{q}(v_0)\langle f(v_0)\rangle$$

$$= \int dv_0 \langle f(v_0)\rangle \frac{d}{dv_0}\left\{\frac{nq^2}{2m}|E|^2 \frac{v_0 \pi}{|k|}\delta(v_0 - v_{ph})\right\}$$

$$= -\frac{\pi\omega_p^2}{|k|}\frac{\omega}{k}\frac{\partial\langle f(v)\rangle}{\partial v}\bigg|_{\omega/k}\left(|E|^2/8\pi\right)$$

$\Rightarrow$

$$Q = -\pi\frac{\omega_p^2}{|k|}\frac{\omega}{k}\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k}\left(|E|^2/8\pi\right)$$

→ agrees with previous result

→ establishes Landau damping mechanism as collisionless heating due secularity at wave-particle resonance.

→ Fate of energy :

$$\frac{\partial W_k}{\partial t} + \cancel{\nabla \cdot S_k} + Q_k = 0$$

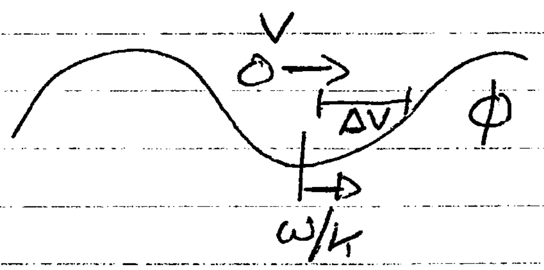
$$\frac{\partial W_k}{\partial t} = -Q_k$$ $\Rightarrow$ L.D. $\leftrightarrow$ wave energy dissipated

at clearly resonant particles heated

so $\frac{\partial \, RPKED}{\partial t} + \frac{\partial}{\partial t} W_k = 0$

$\therefore$ Landau damping heats resonant piece of distribution at expense of wave energy.

→ Clearly, linear theory of Landau damping only valid for times less than bounce time in trough of wave:



$\Delta V \sim \sqrt{q\phi/m}$

$1/\tau_b = k\,\Delta V$

Then $\gamma_k = \gamma_k^{(0)}$ for $t < \tau_b$, only.

> Formal Theory of Landau Damping

Consider initial value problem:

$$f(t=0) = \langle f(v) \rangle + \tilde{f}(0, v, x)$$

Evolution of ϕ ?

(i.) Landau Solution

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\nabla^2 \tilde{\phi} = -4\pi n_0 q \int \tilde{f} dv$$

$$\frac{\partial \tilde{f}_k}{\partial t} + ikv \tilde{f}_k = ik \tilde{\phi}_k \frac{q}{m} \frac{\partial \langle f \rangle}{\partial v}$$

$$k^2 \tilde{\phi}_k = 4\pi n_0 q \int \tilde{f}_k dv$$

Laplace Transform: $\displaystyle \phi_{k,\omega} = \int_0^\infty e^{i\omega t} \phi_k(t)$

$\mathrm{Im}\ \omega > 0$

$$\phi_k(t) = \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} e^{-i\omega t} \phi_{k,\omega} \frac{d\omega}{2\pi}$$

hen:
$$\int_0^\infty e^{i\omega t}\frac{\partial \tilde{f}_k}{\partial t} = -\tilde{f}_k(v,0) - i\omega\int_0^\infty e^{i\omega t}\tilde{f}_k$$
$$= -\tilde{f}_k(v,0) - i\omega\,\tilde{f}_{k,\omega}$$

$$-\tilde{f}_k(v,0) - i(\omega - kv)\tilde{f}_{k,\omega} = i\frac{q}{m}k\,\tilde{\phi}_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

$$\tilde{f}_{k,\omega} = i\frac{\tilde{f}_k(v,0)}{(\omega - kv)} - \frac{q}{m}\frac{k}{(\omega - kv)}\phi_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

$\therefore$

$$k^2\phi_{k,\omega} = 4\pi n_0 q\int dv\left\{-\frac{q}{m}\frac{k}{\omega - kv}\partial\langle f\rangle/\partial v\,\phi_{k,\omega} + i\frac{\tilde{f}_k(v,0)}{\omega - kv}\right\}$$

$\Rightarrow$
$$\epsilon(k,\omega)\phi_{k,\omega} = \frac{4\pi n_0 q i}{k^2}\int dv\,\frac{\tilde{f}_k(v,0)}{\omega - kv}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv\,\frac{\partial\langle f\rangle/\partial v}{\omega - kv}$$

$$\therefore \quad \phi_{k,\omega} = \frac{4\pi n_0 q}{k^2 \epsilon(k,\omega)}\, i \int dv \frac{\tilde{f}_k(v,0)}{\omega - kv}$$

Then,

$$\phi_k(t) = \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\omega \, \frac{4\pi n_0 q}{k^2 \epsilon(k,\omega)} \left(i \int dv \frac{\tilde{f}_k(v,0)}{\omega - kv} \right) e^{-i\omega t}$$

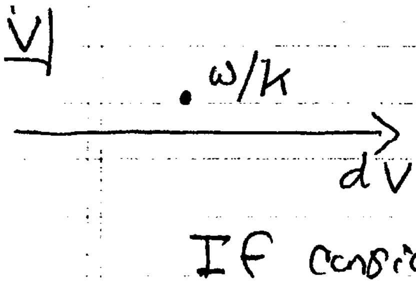
$\phi_k(t)$ determined by analytic structure of integrand

$\Rightarrow$ Singularities $\displaystyle\int dv \frac{\tilde{f}_k(v,0)}{\omega - kv}$

$\Rightarrow \left\{ \begin{matrix} \text{zeroes} \\ \text{singularities} \end{matrix} \right\} \epsilon(k,\omega)$

Now: $\rightarrow \omega = \omega + i\epsilon \Rightarrow v = v - i\epsilon$

$\therefore$ so v integration along contour below pole at ω/k

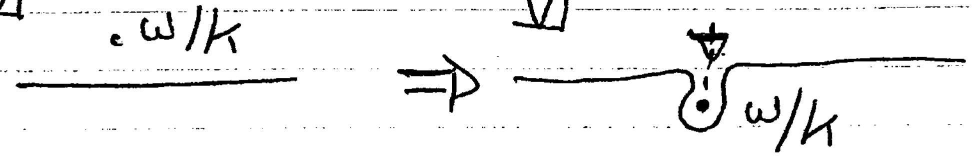


If consider case of damped mode,

analytically continue by deforming contour so pole above ct

V V

i.e.



∴ → singularities $\int dv\, \tilde{f}_k(v,0) \Big/ (\omega - kv) \Big|_{\text{analytic continuation}}$

<u>only</u> at singularities $\tilde{f}_k(v,0)$

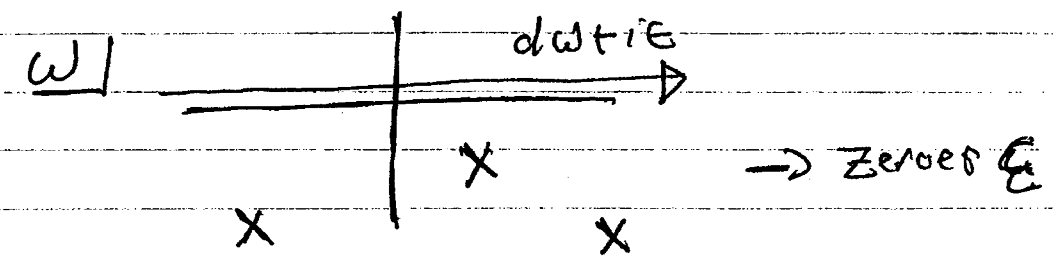
∴

→ assuming $\tilde{f}_k(v,0)$ entire function (no singularity at finite v) and normalizable

∴ $$\int dv \, \frac{\tilde{f}_k(v,0)}{\omega - kv} \to \text{entire function } \omega$$

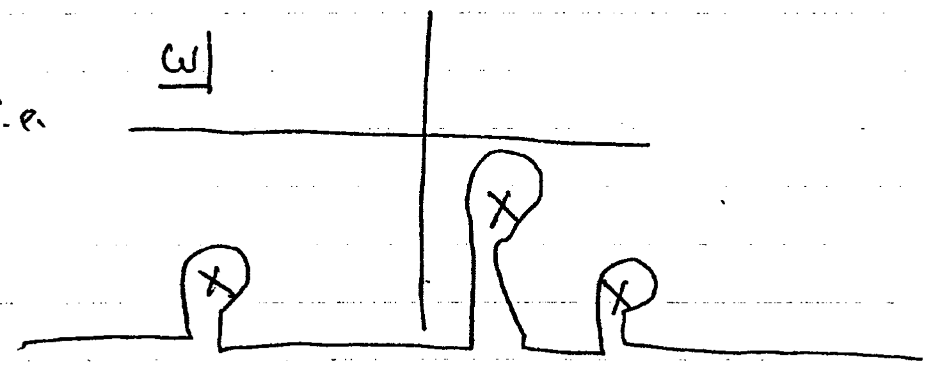
$$\epsilon(k,\omega) \to \text{entire function (same argument)}$$

∴ only singularities of integrand at zeroes $\epsilon(k,\omega)$



⇒ deform ω contour downward till encircles zeroes.

i.e.



Then;

$$\phi_k(t) = \sum_j \phi_k^j e^{-i\omega_{k,r}^j t} e^{-\omega_{k,Im}^j t}$$

↳ residue of j th mode

$\underline{\underline{\text{So}}}$ long time response dominated by <u>least</u> <u>damped</u> <u>mode</u>..

iii) Case - Van Kampen Solution (Schematic)

Aside: General solution of IVP

→ determine complete set of normal modes of system

→ evolution as normal modes with IV Data + Normal Mode Evolution

i.e. Plucked string 

→ Fourier series with IVD ⇒ coefficients

→ Laplace Transform

For Vlasov Plasma ↔ – Continuum of Singular Modes of f
– L.D. as phase mixing

For modes:

$$\frac{\partial \tilde{f}_k}{\partial t} + ikv\tilde{f}_k = i\frac{q}{m}k\tilde{\phi}_k\frac{\partial\langle f\rangle}{\partial v}$$

$$k^2\phi_k = 4\pi n_0 q\int \tilde{f}_k\,dv$$

∴

$$\frac{\partial f_k}{\partial t} + ikv f_k = i\frac{\omega_p^2}{k}\frac{\partial\langle f\rangle}{\partial V}\int dv\, f_k(v)$$

$$\Rightarrow \begin{cases} \dfrac{\partial f_k}{\partial t} + ikv f_k = -ik\,\eta(v)\displaystyle\int_{-\infty}^{+\infty} dv'\, f_k(v') \\ \eta(v) = \dfrac{-\omega_p^2}{k^2}\dfrac{\partial\langle f\rangle}{\partial V} \end{cases}$$

$$f_k = f_{k,\omega}\, e^{-i\omega t}$$

$$(V - \omega/k)\, f_{\omega/k}(v) = -\eta(v)\int_{-\infty}^{+\infty} dv'\, f_{\frac{\omega}{k}}(v')$$

$f = f(v, r)$

$$r \equiv \omega/k$$

$$(V - r)\, f_r(v) = -\eta(V)\int_{-\infty}^{+\infty} dv'\, f_r(v')$$

with normalization $\displaystyle\int_{-\infty}^{+\infty} dv\, f_r(v) = 1$

$$f_r(v) = -\frac{P\,\eta(v)}{V-r} + \lambda(v)\,\delta(V-r)$$

i.e. $(V-r)\,\delta(V-r) = 0$

$$1 = \int_{-\infty}^{+\infty} dv \left(-\frac{P\eta(v)}{v-r} + \lambda(r)\,\delta(v-r) \right)$$ Normalization

$$\therefore \lambda(r) = 1 + \int_{-\infty}^{+\infty} dv \, \frac{P\eta(v)}{v-r}$$

so, normal modes f :

$\rightarrow$ $$f_r(v) = -\frac{P\eta(v)}{v-r} + \lambda(r)\,\delta(v-r)$$

$$\lambda(r) = 1 + \int_{-\infty}^{+\infty} dv \, \frac{P\eta(v)}{v-r}$$

$$\eta(v) = -\frac{\omega_p^2}{k^2} \frac{\partial \langle f \rangle}{\partial v}$$

$\rightarrow$ Modes $\begin{cases} \text{undamped} \\ \text{singular} \end{cases}$ $\Rightarrow$ correspond to ballistic modes (particle streams)

$\rightarrow$ Complete, Orthogonal Set (Case Ann. Phys. 7 349 1959)

Can super-pose to show equivalence to Landau solution; Damping via Phase-Mixing i.e. $$\int e^{-v^2/v_t^2} e^{-ikvt} = \int dv\, e^{-\left(\frac{v}{v_t} + \frac{ikt v_T}{2}\right)^2} e^{-k^2 v_t^2 t^2/4}$$

undamped ballistic mode

Mathematical Note:

$$\epsilon = 1 + \frac{\omega_p^2}{k} \int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

$$= 1 - \frac{\omega_p^2}{k v_{th}} \int dv \, \frac{\langle f \rangle}{(\omega - kv)} \left(\frac{v k}{v_{th} k} - \omega + \omega \right)$$

$$= 1 + \frac{\omega_p^2}{(k v_{th})^2} \int dv \langle f \rangle + \frac{\omega}{k} \frac{\omega_p^2}{(k v_{th})^2} \int dv \, \frac{\langle f \rangle}{v - \frac{\omega}{k}}$$

$$= 1 + \frac{1}{k^2 \lambda_D^2} \left(1 + \frac{\omega}{k v_{th}} \int d\varepsilon \, \frac{e^{-\varepsilon^2}}{\varepsilon - \omega / k v_{th}} \right)$$

$$Z(\omega / k v_{th}) = \int d\varepsilon \, e^{-\varepsilon^2} / \varepsilon - \omega / k$$

Plasma Dispersion Function
(Tabulated)